Lecture 1: Firm and Plant Dynamics

Economics 522

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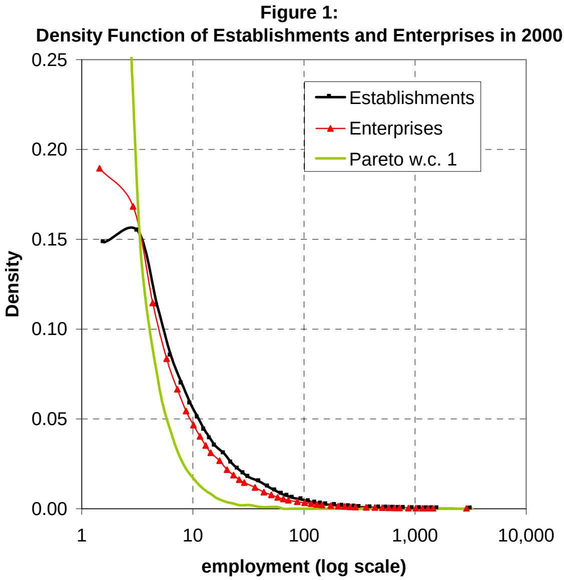
Firm Dynamics and the Size Distribution of Firms



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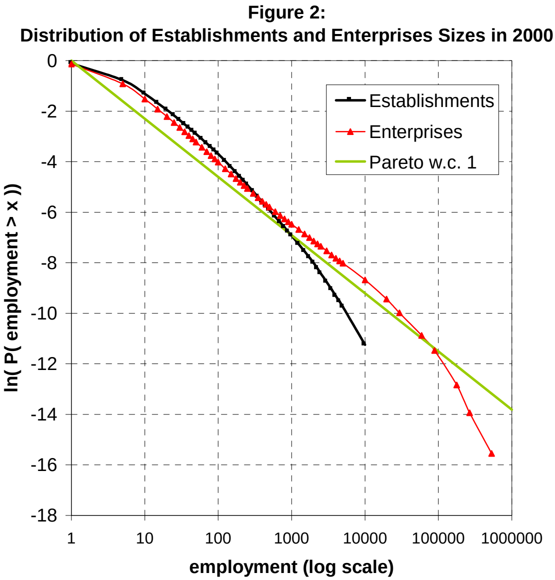


Simon and Bonini (1958)

- Constant returns to scale firms. Can grow arbitrarily large.
- Each employee hires new employee at rate λ per unit of time
- Firms transition from n to $n+1$ at rate λn per unit of time
- New firms enter with $n = 1$ at a rate $(\gamma - \lambda) \sum_{n=1}^{\infty} n M_t(n)$ per unit of time
- $M_n(t) =$ measure of firms of size n at time t
- Then the invariant distribution is a Yule distribution, namely,

$$P_n = \frac{\gamma}{\lambda} \frac{\Gamma(n)\Gamma\left(1 + \frac{\gamma}{\lambda}\right)}{\Gamma\left(n + 1 + \frac{\gamma}{\lambda}\right)}$$

Simon and Bonini (1958)

- Observe that since $\Gamma(n) = (n-1)\Gamma(n-1)$ and $\Gamma(1) = 1$,

$$\lim_{\lambda \uparrow \gamma} P_n = \frac{1}{n(n+1)}$$

and so

$$\sum_{k=n}^{\infty} \frac{1}{k(k+1)} = \frac{1}{n}$$

- So in the limit the distribution is Pareto.

Lucas (1978)

- Team of a manager with skill z with n workers:
- produce $zA(n)$
- decreasing returns to n, so $A' > 0$ and $A'' < 0$
- skill distribution $P(z)$
- For example, $A(n) = n^{\beta}$ with $\beta < 1$, $P(z) = 1 - z^{-\alpha}$
- Then

  $$z = \frac{w}{\beta} n^{1-\beta}$$

  and so

  $$\Pr\left[N(z) \geq n\right] \propto n^{-\alpha(1-\beta)}$$

- Size distribution of firms reflects skill distribution of managers.
  - If skill distribution is Pareto firm distribution is Pareto with coefficient $\alpha\left(1-\beta\right)$

Chatterjee and Rossi-Hansberg (2009)

- Innovation and firm-size dynamics
  - Innovators sometimes sell their ideas to existing firms
  - Or sometimes start a new firm to exploit their idea
- A theory of these decisions
- Private information on the expected return of a new idea
  - High-return ideas induce innovators to set up new firms to exploit the idea
  - Lower-return ideas are sold to existing firms at a price that is not contingent on private information
- Adverse selection as a determinant of firm entry and growth

Introduction

- New firms start with the best ideas
- Prusa and Schmitz (1994) argue that this is the case in the PC software industry
  - Unit sales of the first product of a firm is, on average, 1.86 times the mean unit sales of products in its cohort
  - Unit sales of the second product is only 0.91 times the mean unit sales of products in its cohort
  - The first product is also about twice as successful as the third, fourth, and fifth products
- This is consistent with our theory

Results

- Workers as innovators
- Lesser quality ideas are sold because spinning off is costly
  - Spinoffs lose the option of spinning off in the future with an even better idea
  - Alternatively, spinoffs must pay a start-up cost
- Quality of ideas put into production by buying firms is independent of firm size
  - Expected return on an idea is same for all firms and is equal to the price of the idea.
- This process can generate realistic firm-size distributions

Workers and Entrepreneurs

- Each individual has a unit of time
- Preferences

$$U(\{c_t\}) = E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

  - u is linear or exponential
- Individuals can choose to be Entrepreneurs or Workers
- A worker receives $w > 0$ and gets ideas with probability $\lambda > 0$
- An entrepreneur owns and manages $N \geq 1$ projects, receives profits $\pi(S, N)$ and learns of an idea for sale with probability $\gamma(\lambda, N) > 0$
  - $\pi(S, N) = N\left(S - w\right)$, where $S = \frac{1}{N}\sum_{i=1}^{N} P_i$ and $P_i > 0$ is the per period output from project i
- Individuals consume their income each period

Ideas

- An idea is a non-replicable technology to produce goods using one unit of labor
- Once output is known it becomes a project
- μ is the expected P of an idea and is observed only by the worker who gets the idea
- $\mu \sim H(\mu)$, $P \sim F_\mu(P)$, where $\mu = \int P dF_\mu(P)$
- $\int f(P) dF_\mu(P)$ is increasing in μ for all increasing functions f
- P can be discovered by running the project for one period

Spinoffs and the Market for Ideas

- A worker with an idea μ has two choices
- Sell the idea at the market price $Z > 0$ to an entrepreneur
  - Reveals the mean payoff to the entrepreneur after the entrepreneur buys it
- Start a firm with this idea: a spin-off
  - Discover P by running the project for one period
  - Decide to become an entrepreneur or return to being a worker
  - As an entrepreneur his access to new ideas will be limited to those that are sold in the market
- P is specific to the individual (entrepreneur or worker) who implements the idea
- P-contingent contracts between entrepreneurs and workers are costly and so not used

An Entrepreneur's Problem

- Consider an entrepreneur with average revenue S, projects N, who owns a new idea with mean payoff μ
- If he tests the idea then

$$
\begin{aligned}
V(\mu, S, N) &= \int \left[u\left(\pi(S, N) + w + P - Z - w\right)\right] dF_{\mu}(P) \\
&\quad + \beta \int \max \left[W\left(\frac{NS + P}{N+1}, N+1\right), W(S, N)\right] dF_{\mu}(P)
\end{aligned}
$$

- The continuation value $W(S, N)$ is given by

$$
\begin{aligned}
& W(S, N) \\
= \;& \gamma(\lambda, N) \int^{\mu_H} \max \left[\begin{array}{c} V(\mu, S, N), \\ u(\pi(S, N) - Z + w) + \beta W(S, N) \end{array} \right] dH(\mu) \\
& + (1 - \gamma(\lambda, N)) \left[u\left(\pi(S, N) + w\right) + \beta W(S, N)\right]
\end{aligned}
$$

An Entrepreneur's Value Functions

Lemma

$W(S, N)$ is strictly increasing in S

Lemma

$V(\mu, S, N)$ is strictly increasing in μ and S, and continuous in μ and S

- Let $\mu_L(S, N)$ be the value of μ that solves

$$V(\mu_L, S, N) = u(\pi(S, N) - Z + w) + \beta W(S, N)$$

- An entrepreneur will test the idea if $\mu > \mu_L(S, N)$

A Workers Problem

- Consider a worker with an idea μ
- If he spins off then

$$V_0(\mu) = \int u(P)\, dF_\mu(P) + \beta \int \max\left[W(P,1), W_0\right] dF_\mu(P)$$

- The continuation value W_0 is given by

$$\begin{aligned} W_0 &= \lambda \int \max\left[V_0(\mu), u(w+Z) + \beta W_0\right] dH(\mu) \\ &\quad + (1-\lambda)\left[u(w) + \beta W_0\right] \end{aligned}$$

Worker's Value Functions

Lemma

$V_0(\mu)$ is continuous and strictly increasing in μ

- Let μ_H be the value of μ that solves

$$V_0(\mu_H) = u(w + Z) + \beta W_0$$

- A worker will spin off with the idea if $\mu > \mu_H$

Project Selection

- Let $P_L(N, S)$ solve

$$W\left(\frac{NS + P_L(N, S)}{N+1}, N+1\right) = W(S, N)$$

- An entrepreneur keeps the project if $P \geq P_L(N, S)$
- Let P_H solve

$$W(P_H, 1) = W_0$$

- A spin-off keeps the project if $P \geq P_H$

Equilibrium in the Market for Ideas

- Let $\gamma(\cdot) = \theta\tilde{\gamma}(\lambda, N)$, then θ needs to be such that the supply of ideas and the demand for ideas equalize in equilibrium at price Z
- Let δ_N denote the share of firms of size N in equilibrium
- Market clearing in the market for ideas implies

$$\lambda \sum_{N=1}^{\infty} (N-1)\delta_N = \theta \sum_{N=1}^{\infty} \tilde{\gamma}(\lambda, N)\delta_N$$

- Below we will let $\tilde{\gamma}(\lambda, N) = \lambda N$ and so

$$\theta = 1 - \frac{1}{\nu}$$

  where ν is average firm size in the invariant distribution
- Given $\tilde{\gamma}(\cdot)$ a long-run equilibrium for this economy exists and is unique

Characterization

Theorem

If $u(c_t) = c_t$,

- $P_L(S, N) = w$
- $\mu_L(S, N) = \mu_L < w$
- $P_H = w + f_0,\ f_0 > 0$
- $\mu_H > \mu_L$
- $Z = \frac{1}{H(\mu_H)} \int_{\mu_L}^{\mu_H} \left[\left[\mu - w + \frac{\beta}{1-\beta} \int \max\left[P - w, 0\right] dF_\mu(P) \right] \right] dH(\mu)$

Why?

- μ_L is determined by

$$\int (P-w)\, dF_{\mu_L}(P) + \frac{\beta}{1-\beta} \int_w (P-w)\, dF_{\mu_L}(P) = 0$$

- μ_H is determined by

$$\int (P-w)\, dF_{\mu_H}(P) + \frac{\beta}{1-\beta} \int_{w+f_0} (P-w-f_0)\, dF_{\mu_H}(P) = Z$$

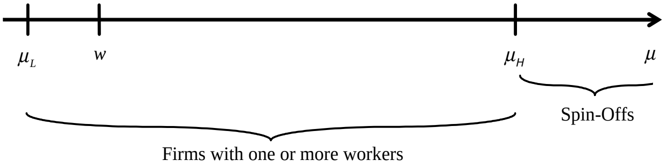
- f_0 is given by

$$\begin{aligned} f_0 &= \\ \lambda \int_{\mu_H} &\left[\int (P-w)\, F_\mu(P) + \frac{\beta}{1-\beta} \int_{w+f_0} [P-w-f_0]\, dF_\mu(P) \right] dH(\mu) \\ +(1-\lambda)Z & \end{aligned}$$

Characterization

- The threshold μ_L is independent of S and N
- The competitive market for ideas is key for this result
  - If entrepreneurs obtain a surplus from ideas then, depending on $\gamma(N, \lambda)$, entrepreneurs with more projects may have greater incentives to test ideas
- $\mu_L < \mu_H$, consistent with the evidence for the software industry in Prusa and Schmitz (1994)
  - Sales of the first product are about twice that of subsequent products

Characterization



Exponential Utility

- In the appendix, we show that all results, except $\mu_L < w$, hold when $u(c_t) = -ae^{-bc_t}$
- The price of ideas is given by

$$Z = \frac{1}{b} \log \left[\frac{1 + \frac{\beta}{1-\beta} \int_{\mu_L}^{\mu_H} \int_{P_L} \left(1 - e^{-b(P-w)}\right) dF_\mu(P) dH(\mu)}{1 - \int_{\mu_L}^{\mu_H} \int \left(1 - e^{-b(P-w)}\right) dF_\mu(P) dH(\mu)} \right] > 0$$

Spinoffs, Firm Growth and Gibrat's Law

- $\gamma(\lambda, N) \propto \lambda N$ or $\gamma(\lambda, N) = \theta\lambda N$
- The transition probabilities of a firm of size N are given by

$$p\left(N, N'\right) = \begin{cases} 0 & \text{for } N' > N+1 \\ \underbrace{\theta\,\lambda \int_{\mu_L}^{\mu_H} \left(1 - F_\mu\left(P_L\right)\right) dH\left(\mu\right) N}_{\lambda_L N} & \text{for } N' = N+1 \\ 1 - \lambda_L N & \text{for } N' = N \\ 0 & \text{for } N' < N \end{cases}$$

- The expected growth rate of employment is independent of firm size (Gibrat's Law)

$$\begin{aligned} g_N\left(N\right) &= \frac{\left(N+1\right) N\lambda_L + N\left(1 - N\lambda_L\right) - N}{N} \\ &= \lambda_L \end{aligned}$$

Growth of Employment in Existing Firms

- λ_L is the ratio of expected number of new workers in existing firms to total employment

$$\begin{aligned} & \sum_{N=1}^{\infty} p\left(N, N+1\right) \cdot \left[E_t \delta_t \left(N\right)\right] \\ = & \sum_{N=1}^{\infty} \lambda\theta \int_{\mu_L}^{\mu_H} \left(1 - F_\mu\left(P_L\right)\right) dH\left(\mu\right) \cdot \left[N E_t \delta_t \left(N\right)\right] \\ \equiv & \lambda_L L_t \end{aligned}$$

Growth of Employment in New Firms

- Expected number of new firms

$$
\begin{aligned}
& \lambda \int_{\mu_H} \left(1 - F_\mu\left(P_H\right)\right) dH\left(\mu\right)\left[NE_t \delta_t\left(N\right)\right] \\
\equiv \quad & \lambda_H L_t
\end{aligned}
$$

- λ_H is the ratio of number of workers in new firms to total employment

Distribution of Employment Shares

- For E_t large, $L_{t+1} = (1 + \lambda_H + \lambda_L)\, L_t$ and $E_{t+1} = E_t + \lambda_H L_t$
- Let ϕ_N denote the probability that a worker is employed by a firm with N workers
- The invariant distribution of employment shares solves

$$\begin{aligned} \left[\phi_1 (1 - \lambda_L) + \lambda_H\right] L &= \phi_1 (1 + \lambda_L + \lambda_H)\, L \\ &\Rightarrow \phi_1 = \frac{1}{1 + 2(\lambda_L / \lambda_H)} \end{aligned}$$

and

$$\phi_N (1 - \lambda_L N) + \phi_{N-1} \lambda_L (N-1) + \phi_{N-1} \lambda_L = \phi_N (1 + \lambda_L + \lambda_H)$$

$$\Rightarrow \phi_N = \phi_{N-1} \frac{(\lambda_L / \lambda_H)\, N}{1 + (\lambda_L / \lambda_H) / (N+1)}$$

- It is easy to show that $\sum_{N=1}^{\infty} \phi_N = 1$

Existence and Uniqueness of Invariant Distributions

Theorem

There exists a unique invariant distribution ϕ of employment shares across firms sizes

- The invariant distribution of firm sizes, δ_N, is

$$\delta_N = \frac{\phi_N}{N \sum_{N=1}^{\infty} \frac{\phi_N}{N}}$$

  Clearly, since $\sum_{N=1}^{\infty} \phi_N = 1$, $0 < \sum_{N=1}^{\infty} \frac{\phi_N}{N} < 1$ and so δ_N is well defined, exists, and is unique

Corollary

There exists a unique invariant distribution δ of firm sizes

- ϕ and δ only depend only on (λ_H / λ_L)

How Close to Pareto?

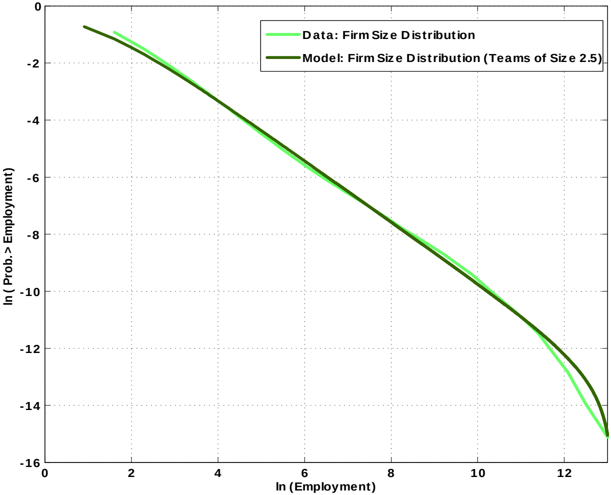
$$\phi_N = \phi_{N-1} \frac{\lambda_L N}{\lambda_H + \lambda_L (N+1)}$$

Lemma

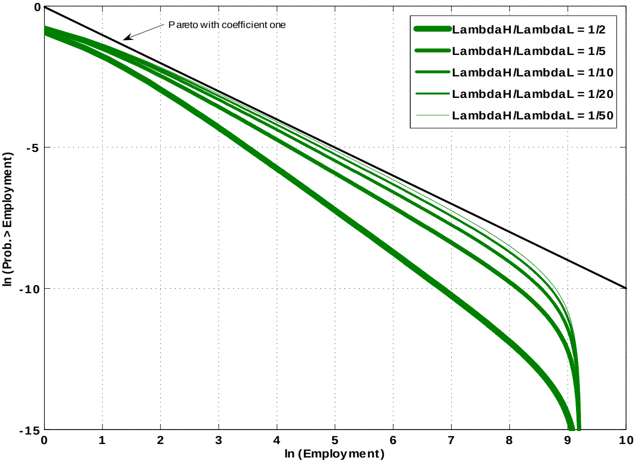
Simon and Bonini (1958). As $N \to \infty$, the density of firm sizes is arbitrarily close to the density of a Pareto distribution with coefficient one. Furthermore, the distribution of firm sizes is closer to a Pareto distribution with coefficient one, the smaller the mass of workers in new firms, λ_H

Comparison With Data

- Data: (SBA)
  - λ_H/λ_L can be measured as number of net new workers in new firms vs. net new workers in old firms
  - From 1989 to 2003: $\lambda_H/\lambda_L = 0.0736$ (or 0.1235 if averaged year by year)
- Truncate distribution at $N = 500000$
- Size of Spinoffs is 2.5 instead of 1



Data: Firm Size Distribution
Model: Firm Size Distribution (Teams of Size 2.5)
ln (Prob.> Employment)
ln (Employment)
0
-2
-4
-6
-8
-10
-12
-14
-16
0
2
4
6
8
10
12


Pareto with coefficient one
LambdaH/LambdaL = 1/2
LambdaH/LambdaL = 1/5
LambdaH/LambdaL = 1/10
LambdaH/LambdaL = 1/20
LambdaH/LambdaL = 1/50
ln (Prob.> Employment)
ln (Employment)
0
-5
-10
-15
0 1 2 3 4 5 6 7 8 9 10

Summary

- A private-information-based theory of innovation, entry and firm growth
- High quality ideas engender in spinoffs while lesser quality ideas engender growth of existing firms
- Market for ideas implies that firm behavior, μ_L and P_L,is independent of (S, N) regardless of $\gamma(N, \lambda)$
- If $\gamma(\lambda, N) \propto \lambda N$, the invariant distribution of firm sizes is Pareto w.c. 1 in the upper tail

Klette and Kortum (2004)

- Stylized facts:
    1. Productivity and R&D across firms are positively related, whereas productivity growth is not strongly related to firm R&D.
    2. Patents and R&D are positively related both across firms at a point in time and across time for given firms.
    3. R&D intensity is independent of firm size.
    4. The distribution of R&D intensity is highly skewed and a considerable fraction of firms report zero R&D.
    5. Differences in R&D intensity across firms are highly persistent.
    6. Firm R&D investment follows essentially a geometric random walk.
    7. The size distribution of firms is highly skewed.
    8. Smaller firms have a lower probability of survival, but those that survive tend to grow faster than larger firms. Among larger firms, growth rates are unrelated to past growth or to firm size.
    9. The variance of growth rates is higher for smaller firms.
    10. Younger firms have a higher probability of exiting, but those that survive tend to grow faster than older firms. The market share of an entering cohort of firms generally declines as it ages.

Klette and Kortum (2004)

- Firm growth driven by technological innovation.
- Technological innovation driven by firm R&D investment.
- Innovation allows firm to expand its product line.
- As in Simon and Bonini, but unlike Lucas, no natural size of a firm.
- Firm can grow arbitrarily large, although it takes time and luck.
- Firms eventually hit a string of bad luck and exit.

Endogenous Technological Change Model

- Models developed by Aghion and Howitt, Grossman and Helpman, and Romer.
  - Captured idea that technological advances are non rival.
  - Imperfect competition and spillovers support continuing R&D and growth.
- Grossman and Helpman's quality ladders model:
  - Growth via better and better versions of a fixed continuum of goods.
  - Schumpeterian creative destruction.
  - Perfect setting for a better model of innovative firms.

Quality Ladders Model in Aggregate

- Cobb Douglas preferences over unit continuum of goods.

$$\ln C_t = \int_0^1 \ln[x_t(j) z_t(j)] dj$$

- Quality ladder: $z_t(j) = q^{J_t(j)}$, with steps $q > 1$.
- Intertemporal utility:

$$U = \int_0^\infty e^{-\rho t} \ln C_t dt$$

- Aggregate expenditures are numeraire, hence unit flow of spending on each good.

A Firm

- Firm is top step of the ladder for some integer number of goods, n.
- Every firm has unit production cost w.
- Bertrand competition with next step on the ladder.
- Only top step technology is used and $p = wq$.
- Firm's total flow revenue is n.
- Flow profit per good is $\pi = 1 - q^{-1}$.

Innovation

- A size n firm investing in R&D may innovate, at Poisson rate I, and become $n+1$.
- It may lose a good to a competitor, with Poisson hazard μn, and become $n-1$.
- Think of n as measuring firm's knowledge capital.
- Knowledge accumulates for society, but zero-sum game for firms.
- Assume $I = G(R, n)$ where R denotes R&D and I innovation:
  - strictly increasing in R.
  - strictly concave in R.
  - strictly increasing in n.
  - CRS in R and n.
- Implies $R = nc(I/n)$:
  - c twice differentiable, $c(0) = 0$, $c'(0) < \pi/r$, and $[\pi - c(\mu)]/r \leq c'(\mu)$.

R&D Investment

- Firm with no products has no value, $V(0) = 0$.
- Jacobi-Bellman's equation for a firm with $n > 0$ products

$$\begin{aligned} & rV(n) \\ = \; & \max_I \left\{ \pi n - nc(I/n) + I[V(n+1) - V(n)] - \mu n[V(n) - V(n-1)] \right\}. \end{aligned}$$

- Solution: $V(n) = vn$, $I(n) = \lambda n$.
- Satisfying $c'(\lambda) = v$ (for $\lambda > 0$) and $v = [\pi - c(\lambda)]/(r + \mu - \lambda)$.

Implications

- Innovation intensity $\lambda = I(n)/n$ is independent of firm size.
- Satisfies $0 \leq \lambda \leq \mu$, with λ increasing in π.
- Research intensity $R/n = c(\lambda)$ independent of firm size.
- Firm value is sum of value of each product, $V(n) = nv$.
- Firm value is sum of production nv_p and research nv_r divisions:

$$v_p = \pi/(r+\mu), \quad v_r = \frac{\frac{\lambda}{r+\mu}\pi - c(\lambda)}{r+\mu-\lambda}$$

- Knowledge Capital
  - Empirical literature, Griliches (1979), measures knowledge capital as firm's stock of past R&D.
  - The present model provides a rationale, although n is the true knowledge capital.
  - What is the expectation of n given past R&D?

$$E[n_t|\{R_s\}] = E\int_{-\infty}^{t} e^{-\mu(t-s)} I_s ds = a\int_{-\infty}^{t} e^{-\mu(t-s)} R_s ds = aK_t$$

where stock K_t is indicator of knowledge capital.

Firm Dynamics

- Define $p_n(t; n_0)$ as probability firm has n products at date t given n_0 at date 0.
- W.l.o.g., consider firm entering at date 0 with one innovation, $p_n(t) = p_n(t; 1)$.
- Must satisfy a system of equations:

$$\dot{p}_0(t) = \mu p_1(t)$$

and for $n \geq 1$:

$$\dot{p}_n(t) = (n-1)\lambda p_{n-1}(t) + (n+1)\mu p_{n+1}(t) - n(\lambda + \mu) p_n(t)$$

- Define

$$\gamma(t) = \frac{\lambda[1 - e^{-(\mu-\lambda)t}]}{\mu - \lambda e^{-(\mu-\lambda)t}}$$

- For $n = 0$:

$$p_0(t) = \mu \gamma(t) / \lambda$$

- For $n \geq 1$, geometric distribution conditional on survival through date t:

$$\frac{p_n(t)}{1 - p_0(t)} = [1 - \gamma(t)]\gamma(t)^{n-1}$$

Implications

- Note that $\gamma(0) = 0$, $\gamma'(t) > 0$, $\lim_{t \to \infty} \gamma(t) = \lambda/\mu$, $\lim_{\lambda \to \mu} \gamma(t) = \mu t/(1 + \mu t)$.
- Firms eventually die: $\lim_{t \to \infty} p_0(t) = 1$.
- Conditional on survival, the expectation and variance of firm size increases.
- Distribution of age: $\Pr[A \leq a] = p_0(a)$.
- Note $1 - \gamma(a)$ is probability of being in state 1 conditional on survival to age a.
- Hazard rate at age a is $\mu[1 - \gamma(a)]$.
- Firm with n_0 products at date 0 behaves as $n_0$firms of size 1 evolving independently.
- Thus, for example: $p_0(t; n_0) = p_0(t)^{n_0}$.

Firm Growth

- Let N_t be random size of a firm (in terms of sales) at date t.
- Growth since time 0: $G_t = (N_t - N_0)/N_0$.
- Expected growth: $E[G_t|N_0 = n] = e^{-(\mu-\lambda)t} - 1$, Gibrat's Law.
- Limit as $t \to 0$ of $E[G_t|N_0 = n]/t = -(\mu - \lambda)$, but reinterpret negative drift in light of numeraire (measured nominal GDP grows).
- Limit as $t \to 0$ of $Var[G_t|N_0 = n]/t = (\mu + \lambda)/n$, i.e. weak form of Gibrat's Law.
- Conditional on survival:

$$E[G_t|N_t > 0, N_0 = n] = \frac{e^{-(\mu-\lambda)t}}{1 - p_0(t)^n} - 1$$

which is decreasing in n. Selection effect.

Aggregate Accounting

- Let $M_n(t)$ be the measure of size n firms in the economy at date t.
- Total measure of firms is $M(t) = \sum_{n=1}^{\infty} M_n(t)$.
- Accounting identity due to unit continuum of goods: $\sum_{n=1}^{\infty} nM_n(t) = 1$.
- Total innovation rate by incumbent firms:

$$\sum_{n=1}^{\infty} M_n(t) I(n) = \sum_{n=1}^{\infty} M_n(t) \lambda n = \lambda.$$

- If entrants innovate at rate η, then $\mu = \eta + \lambda$.

Entry

- Potential entrants must invest at rate F to obtain a Poisson hazard 1 of entering with a single product.
- Consider an equilibrium with $\eta > 0$ and $\lambda > 0$.
- Freedom to pursue entry implies $F = V(1) = v$.
- From Bellman equation $v = c'(\lambda)$ so $F = c'(\lambda)$, which nails down λ.
- Also, from the Bellman equation,

$$v = \frac{\pi - c(\lambda)}{r + \mu - \lambda} = \frac{\pi - c(\lambda)}{r + \eta}.$$

- Solve for the entry rate

$$\eta = [\pi - c(\lambda)]/F.$$

- In general, two other cases: all innovation done by incumbents or all innovation done by entrants.

Size Distribution

- For $n = 1$:

$$\dot{M}_1(t) = \eta + 2\mu M_2(t) - (\lambda + \mu) M_1(t).$$

- And, for $n \geq 2$:

$$\dot{M}_n(t) = (n-1)\lambda M_{n-1}(t) + (n+1)\mu M_{n+1}(t) - n(\lambda + \mu) M_n(t).$$

- Finally, by our accounting identity, $\dot{M}(t) = \eta - \mu M_1(t)$.
- For stationary distribution, set all time derivatives to zero, drop time subscripts, and solve.

Size Distribution

- Starting with accounting: $M_1 = \eta/\mu$.
- Plug into the $n = 1$ case to get $M_2 = \lambda\eta/[2\mu^2]$.
- Keep going, and by induction, for all $n \geq 1$:

$$M_n = \frac{\lambda^{n-1}\eta}{n\mu^n} = \frac{\theta}{n}\left(\frac{1}{1+\theta}\right)^n$$

  where $\theta = \eta/\lambda$.
- Distribution has a long right tail of large firms when θ is close to zero. In that case some incumbents have time to get very large.
- The total mass of firms is

$$M = \theta \ln \frac{1+\theta}{\theta}$$

  which is large when entry dominates (producing many small size 1 firms).

General Equilibrium

- Labor supply: $L = L_X + L_S + L_R$
  - L_X for good production, L_S for innovation in new firms, L_R innovation in existing firms
- Fixed cost of entry: $F = wh$ (team of h gets first innovation at rate 1).
- Research at incumbents: $c(x) = wl_R(x)$ (takes $l_R(x)$ researchers for size 1 firm to innovate at rate x).
- Stationary equilibrium: constant values of r, w, v, λ, and η such that:
  - potential entrants expect to break even.
  - incumbent firms optimize.
  - representative consumer maximizes utility.
  - labor market clears.
- Consider equilibrium with constant labor allocation and $\eta > 0$, $\lambda > 0$:
  - If $L_S > 0$ then $v = wh$.
  - If $L_R > 0$ then $v = wl'_R(\lambda)$, i.e. $l'_R(\lambda) = h$.

Solution

- Since aggregate profits are π: $wL_X = (1 - \pi)$.
- Entrants: $wL_S = w\eta h = \eta v$.
- Incumbent researchers:

$$L_R = \sum_n M_n n l_R(\lambda) = l_R(\lambda).$$

- Total equity value of all firms:

$$\sum_n M_n V(n) = \sum_n M_n n v = v.$$

- Return on equity

$$rv = \pi - wl_R(\lambda) + \lambda v - \mu v = \pi - wl_R(\lambda) - \eta v$$

Solution

- Accounting for aggregate income $Y = 1$:

$$\begin{aligned} Y &= wL + rv \\ &= wL + \pi - wL_R - wL_S \\ &= wL_X + \pi \end{aligned}$$

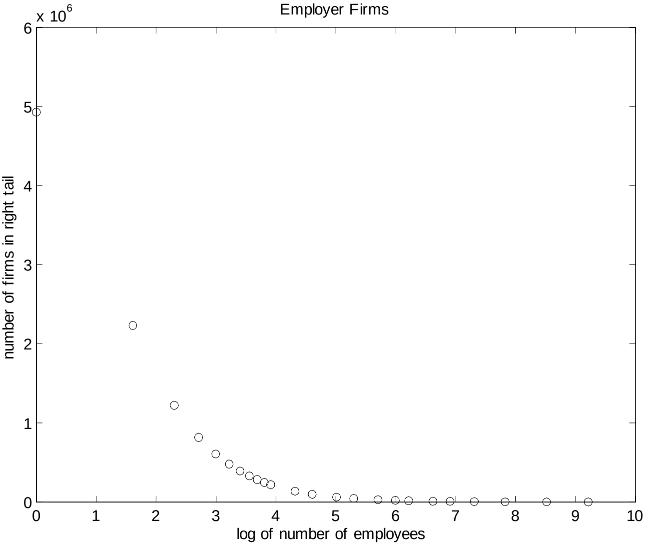
- Willing to accept return on equity if $r = \rho$ (from consumption Euler equation).
- Since $1 = wL + \rho v = wL + \rho wh$ we have

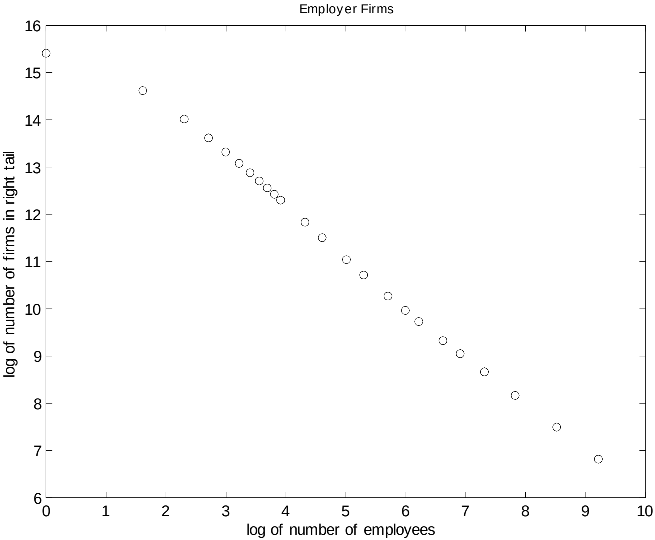
$$w = 1/(L + \rho h)$$

- Thus $L_X = (1 - \pi)(L + \rho h)$.
- From above $L_R = l_R(\lambda)$ is pinned down by $l'_R(\lambda) = h$.

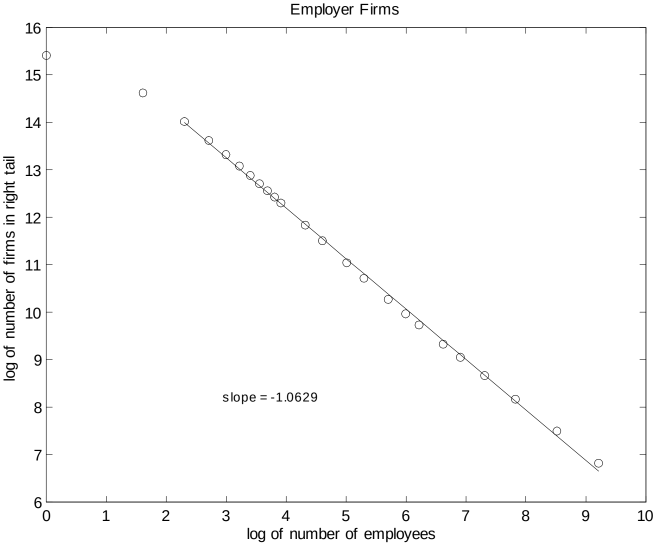
Luttmer (2007)

- Firms are monopolistic competitors
- Permanent shocks to preferences and technologies associated with firms
- Low productivity firms exit, new firms imitate and attempt to enter
  - Selection produces Pareto right tail rather than log-normal.
  - Population productivity grows faster than mean of incumbents.
  - Thickness of right tail depends on the difference.
  - Zipf tail when entry costs are high or imitation is difficult.

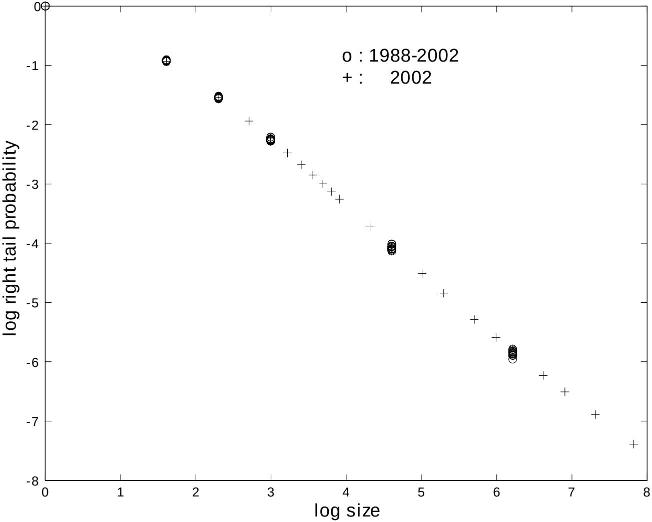

Employer Firms
x 10^6
6
5
4
3
2
1
0
number of firms in right tail
0
1
2
3
4
5
6
7
8
9
10
log of number of employees



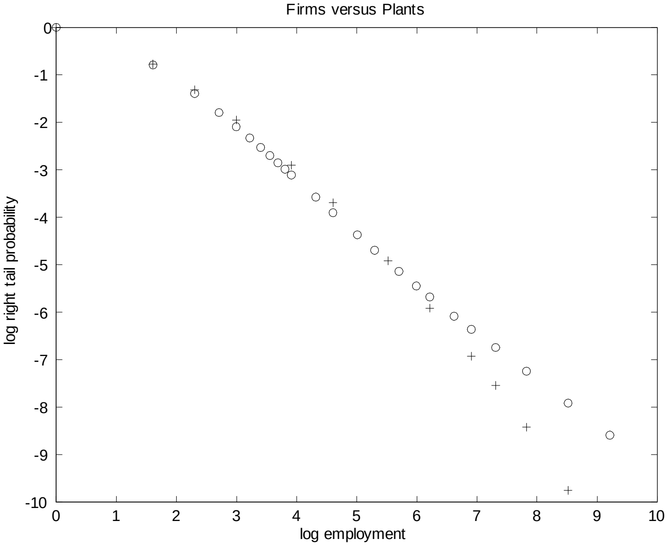
Employer Firms
log of number of firms in right tail
log of number of employees
16
15
14
13
12
11
10
9
8
7
6
0
1
2
3
4
5
6
7
8
9
10

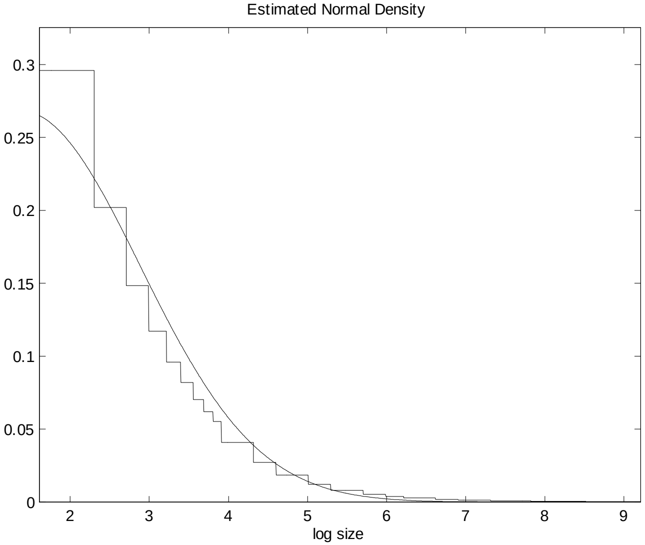


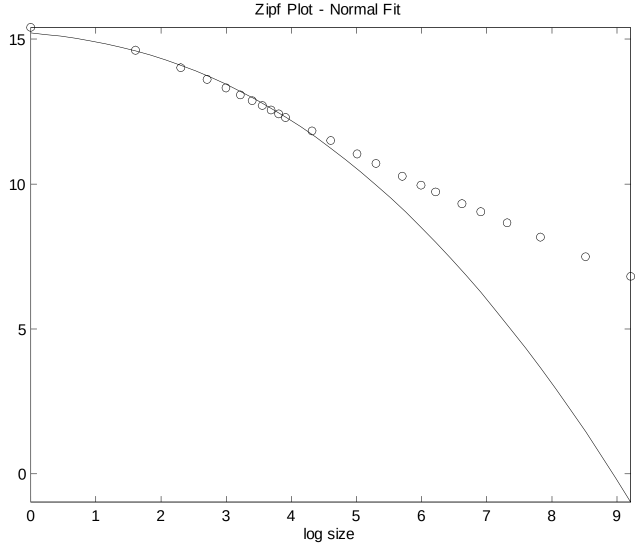
Employer Firms
slope = -1.0629
log of number of firms in right tail
log of number of employees
16
15
14
13
12
11
10
9
8
7
6
0
1
2
3
4
5
6
7
8
9
10



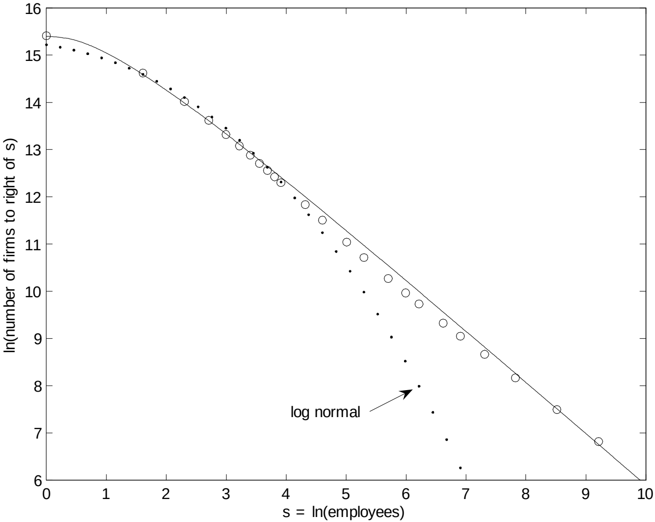
o : 1988-2002
+ : 2002
log right tail probability
log size
0
-1
-2
-3
-4
-5
-6
-7
-8
0
1
2
3
4
5
6
7
8


Firms versus Plants
log right tail probability
log employment
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
0
1
2
3
4
5
6
7
8
9
10


Estimated Normal Density
0.3
0.25
0.2
0.15
0.1
0.05
0
2
3
4
5
6
7
8
9
log size


Zipf Plot - Normal Fit
15
10
5
0
0
1
2
3
4
5
6
7
8
9
log size

This Model



ln(number of firms to right of s)
s = ln(employees)
log normal
16
15
14
13
12
11
10
9
8
7
6
0
1
2
3
4
5
6
7
8
9
10

The Economy

- Preferences:
  - differentiated commodities with permanent taste shocks
- Technologies:
  - at a cost, entrants draw technologies from some distribution
  - fixed overhead labor, asymptotic constant returns to scale
  - random productivity, quality growth.

Consumers

- A population $He^{\eta t}$ with preferences over aggregate consumption:

$$\left(\mathrm{E}\left[\int_0^\infty \rho e^{-\rho t}\left[C_t e^{-\eta t}\right]^{1-\gamma}\mathrm{d}t\right]\right)^{1/(1-\gamma)}$$

- where:

$$C_t = \left[\int u^{1-\beta} c_t^{\beta}(u,p)\mathrm{d}M_t(u,p)\right]^{1/\beta}$$

- Real expenditures are:

$$\begin{aligned}\frac{pc_t(u,p)}{P_t} &= (uC_t)^{1-\beta}\, c_t^{\beta}(u,p),\\ P_t &= \left[\int up^{-\beta/(1-\beta)}\mathrm{d}M_t(u,p)\right]^{-(1-\beta)/\beta}\end{aligned}$$

Firms

- Firm-specific output and technologies.
- Asymptotic constant returns to scale.
- Continuation requires λ_{F} units of labor per unit of time.
- Unit arrival rate of new firms costs λ_{E} units of labor per unit of time.
- Output:

$$y_{t,a} = z_{t,a} A(L_{t,a})$$

- Implied variable profits:

$$\max_{L} \left\{ Z_{t,a}^{\beta} C_{t+a}^{1-\beta} \left[A(L)\right]^{\beta} - w_{t+a} L \right\}$$

where

$$Z_{t,a} = \left(u_{t,a}^{1-\beta} z_{t,a}^{\beta} \right)^{1/\beta}$$

evolves according to the black-box process:

$$Z_{t,a} = Z \exp\left(\theta_{\mathrm{E}} t + \theta_{\mathrm{I}} a + \sigma_{\mathrm{Z}} W_a\right)$$

- The initial condition Z is drawn from some distribution G.

The Growth Rate

Balanced growth:

- wages $w_t = we^{\kappa t}$
- aggregate consumption $C_t = Ce^{(\kappa+\eta)t}$
- the number of firms $M_t = Me^{\eta t}$.
- Distribution of $Z_{t,a}^{\beta} C_{t+a}^{1-\beta} [A(L_{t,a})]^{\beta} - w_{t+a} L_{t,a}$ must have a trend $e^{\kappa t}$
- This yields:

$$\kappa = \underbrace{\theta_{\mathrm{E}}}_{\text{quantity and quality}} + \underbrace{\left(\frac{1-\beta}{\beta}\right)\eta}_{\text{variety}}$$

The Firm-Specific State Variable

- Variable profits:

$$Z_{t,a}^{\beta} C_{t+a}^{1-\beta} \left[A(L_{t,a})\right]^{\beta} - w_{t+a} L_{t,a} = w_{t+a} \left[S_{t,a}^{1-\beta} \left[A(L_{t,a})\right]^{\beta} - L_{t,a} \right]$$

where:

$$S_{t,a} = \left(\frac{Z_{t,a}}{w_{t+a}} \right)^{\beta/(1-\beta)} \frac{C_{t+a}}{w_{t+a}}$$

- Dynamics:

$$S_a = \exp\left(s[Z]\right) \left[\exp\left([\theta_{\mathrm{I}} - \theta_{\mathrm{E}}]a + \sigma_{\mathrm{Z}} W_a\right)\right]^{\beta/(1-\beta)}$$

where:

$$e^{s[Z]} = \left(\frac{Z}{w} \right)^{\beta/(1-\beta)} \frac{C}{w}$$

The Firm-Specific State Variable

- So $s_a = \ln(S_a)$ follows:

$$\mathrm{d}s_a = \mu \mathrm{d}a + \sigma \mathrm{d}W_a$$

  where:

$$\begin{bmatrix} \mu \\ \sigma \end{bmatrix} = \frac{\beta}{1-\beta} \begin{bmatrix} \theta_{\mathrm{I}} - \theta_{\mathrm{E}} \\ \sigma_{\mathrm{Z}} \end{bmatrix}$$

- Typically, $\mu < 0$, but can have $\mu > 0$ if $\eta > 0$.

Variable Profits

- Let $L(s)$ solve:
- $$Q(s) = \max_{L} \frac{1}{\lambda_{\mathrm{F}}} \left\{ [e^s]^{1-\beta} [A(L)]^{\beta} - L \right\}$$
- If $A(L) \sim L$ for large L:
$$L(s) \sim e^s \text{ for } s \text{ large}$$
- Need also:
- $$Q(s) \to 0 \text{ for } s \text{ small}$$
- to guarantee exit of low-productivity firms.

The Stopping Problem

- The value of a firm with productivity $Z_{t,a}$ at time $t+a$ is:

$$w_{t+a}\lambda_{\mathrm{F}} V\left(s\left[Z_{t,a}e^{-\theta_{\mathrm{E}}t}\right]\right)$$

where:

$$V(s) = \max_{\tau} \mathrm{E}\left[\int_0^{\tau} e^{-(r-\kappa)a}\left[Q(s_a)-1\right]\mathrm{d}a \middle| s_0 = s\right]$$

- The Bellman equation is ($\mathcal{A}$ = Apply Ito):

$$rV(s) = \kappa V(s) + \mathcal{A}V(s) + Q(s) - 1$$

- At the exit barrier b:

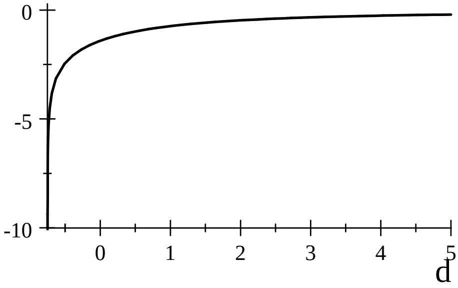
$$V(b) = 0$$

- The exit barrier must be such that:

$$\mathrm{D}V(b) = 0$$

The Exit Barrier with A(L) = L

- Log of profitability $q = \ln[Q(b)]$ at exit, as a function of drift:



where $\mathbf{d} = -\mu/(\sigma^2/2)$ and $[-\mu, \sigma] = [\theta_E - \theta_I, \sigma_Z]\beta/(1-\beta)$.

- Faster aggregate productivity growth: firms "throw in the towel" more quickly.

Entry

- Labor cost of an arrival rate of I_t entry opportunities per unit of time:

$$L_{E,t} = \lambda_{\mathrm{E}} I_t$$

- An entry opportunity yields a draw Z from a distribution G.
- Zero-profit condition:

$$\lambda_{\mathrm{E}} = \lambda_{\mathrm{F}} \int V(s[Z]) G(\mathrm{d}Z)$$

- Technology adoption: G exogenous.

Kolmogorov Forward Equation

- $$y_{t+h} = y_t + \begin{cases} \mu h + \sigma\sqrt{h} & \text{w.p. } \frac{1}{2} \\ \mu h - \sigma\sqrt{h} & \text{w.p. } \frac{1}{2} \end{cases}$$

- Let $f(t, y)$ be the density at time t:

$$f(t+h, y) = \frac{1}{2} f\left(t, y - \mu h - \sigma\sqrt{h}\right) + \frac{1}{2} f\left(t, y - \mu h + \sigma\sqrt{h}\right)$$

- Therefore:

$$\frac{1}{h}\left[f(t+h, y) - f(t, y)\right] = \frac{1}{h}\left[f(t, y - \mu h) - f(t, y)\right] +$$
$$\frac{\frac{1}{2}\sigma^2}{\left(\sigma\sqrt{h}\right)^2}\left[f\left(t, y - \mu h - \sigma\sqrt{h}\right) - 2f(t, y - \mu h) + f\left(t, y - \mu h + \sigma\sqrt{h}\right)\right]$$

- Taking limits:

$$\mathrm{D}_t f(t, y) = -\mu \mathrm{D}_y f(t, y) + \frac{1}{2}\sigma^2 \mathrm{D}_y^2 f(t, y)$$

Exit Rates

- Suppose:

$$\mathrm{d}y_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t$$

  together with an exit barrier at b, so that $f(t, b) = 0$.
- Measure of a cohort:

$$m(t) = \int_b^\infty f(t, y) \mathrm{d}y$$

- Then, using integration-by-parts twice:

$$\begin{aligned} \mathrm{D}m(t) &= \int_b^\infty \mathrm{D}_t f(t, y) \mathrm{d}y = \int_b^\infty \left[-\mu \mathrm{D}_y f(t, y) + \frac{1}{2} \sigma^2 \mathrm{D}_{yy} f(t, y) \right] \mathrm{d}y \\ &= -\frac{1}{2} \sigma^2 \mathrm{D}_y f(t, b) \end{aligned}$$

Firm Population Dynamics

- Density of firms:

$$k(t,a,s) = m(a,s) I e^{\eta t}$$

- Kolmogorov:

$$\mathrm{D}_t k(t,a,s) = -\mathrm{D}_a k(t,a,s) - \mu \mathrm{D}_s k(t,a,s) + \frac{1}{2}\sigma^2 \mathrm{D}_{ss} k(t,a,s)$$

Therefore:

$$\mathrm{D}_a m(a,s) = -\eta m(a,s) - \mu \mathrm{D}_s m(a,s) + \frac{1}{2}\sigma^2 \mathrm{D}_{ss} m(a,s)$$

- At age zero:

$$\lim_{a \downarrow 0} \int_b^s m(a,x) \mathrm{d}x = F(s) - F(b)$$

- where $G(Z) = F(s[Z])$.
- At the exit boundary, $m(a,b) = 0$.

Firm Population Dynamics

- Then

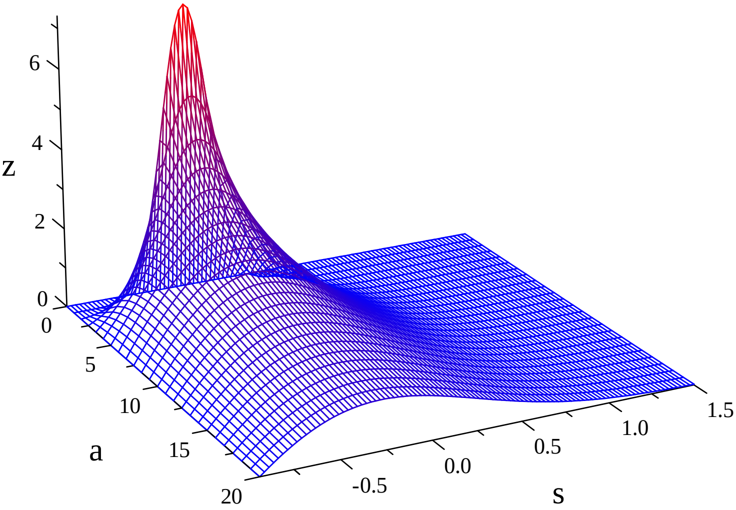
$$m(a,s) = \int_b^\infty e^{-\eta a}\psi(a,s|x)F(\mathrm{d}x)$$

where:

$$\psi(a,s|x) = \frac{1}{\sigma\sqrt{a}}\left[\phi\left(\frac{s-x-\mu a}{\sigma\sqrt{a}}\right) - e^{-\frac{\mu(x-b)}{\sigma^2/2}}\phi\left(\frac{s+x-2b-\mu a}{\sigma\sqrt{a}}\right)\right]$$

- and where ϕ is the standard normal probability density.
- $\psi(a,s|x)$ is the density of survivors at age a with profitability s of the cohort that entered with the same initial profitability x (not a p.d.f.)

The Life of a Cohort



The Size Marginal

- Integrating over age gives:

$$m(s) = \int_b^\infty \pi(s|x) \left(\frac{1 - e^{-\zeta_*(x-b)}}{\eta} \right) F(\mathrm{d}x)$$

where

$$\begin{aligned} \pi(s|x) &= \zeta e^{-\zeta(s-b)} \left(\frac{e^{\zeta_*(x-b)} - 1}{\zeta_*} \right)^{-1} \\ &\quad \times \min \left\{ \frac{e^{[\zeta+\zeta_*](s-b)} - 1}{\zeta + \zeta_*}, \frac{e^{[\zeta+\zeta_*](x-b)} - 1}{\zeta + \zeta_*} \right\} \end{aligned}$$

and

$$\begin{aligned} \zeta &= -\frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}} \\ \zeta_* &= \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}} \end{aligned}$$

The Power Law

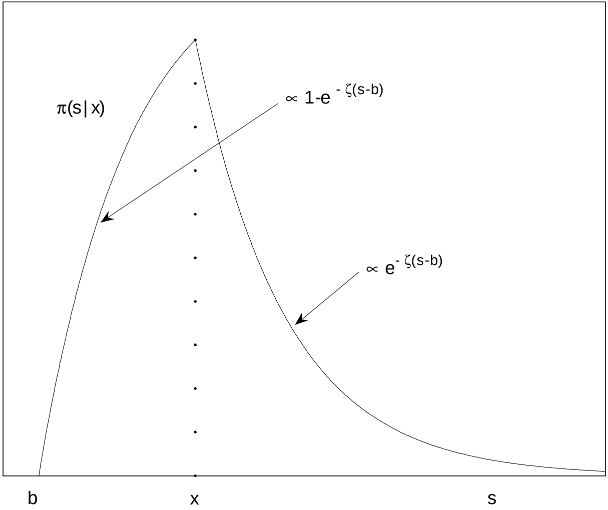
- The size marginal is a weighted average of:
- $$\int_0^\infty e^{-\eta a}\psi(a,s|x)\mathrm{d}a \quad \propto \quad e^{-\zeta(s-b)}\left(\min\left\{e^{[\zeta+\zeta_*](s-b)}, e^{[\zeta+\zeta_*](x-b)}\right\}-1\right)$$
- If $\eta = 0$ then $\zeta_* = 0$ and

$$\zeta \quad = \quad -\frac{\mu}{\sigma^2/2} \quad = \quad \frac{\theta_{\mathrm{E}} - \theta_{\mathrm{I}}}{\frac{1}{2}\left(\frac{\beta}{1-\beta}\right)\sigma_{\mathrm{Z}}^2}$$

where

$$\theta_{\mathrm{E}} \quad = \quad \text{growth rate in population}$$
$$\theta_{\mathrm{I}} \quad = \quad \text{growth rate among incumbents}$$

Stationary Size Density



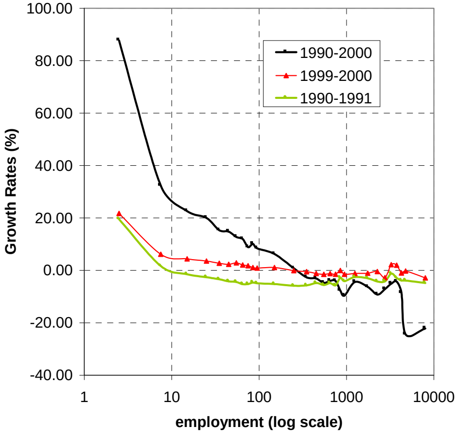
Rossi-Hansberg and Wright (2007)

- To what extent do establishment dynamics and the size distribution of establishments reflect the efficiency of resource allocation?
- Any theory of establishment growth must be consistent with the robust set of stylized facts on scale dependence in establishment dynamics
- In this paper we present a theory of establishment size dynamics where establishment heterogeneity is the result of industry heterogeneity
- The efficient accumulation of industry specific human capital rationalizes the set of stylized facts
    - Mean reversion in the stock of specific human capital drives mean reversion in establishment sizes, which is reflected in the size distribution
- Our theory also uncovers novel relationships between technology and establishment dynamics that we document with a new data set

Facts on Establishments

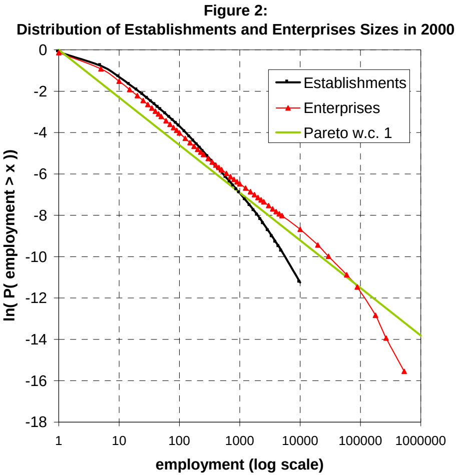
- Small establishments grow faster than large establishments

Figure 3: Establishment Conditional Growth Rates, 1990-2000



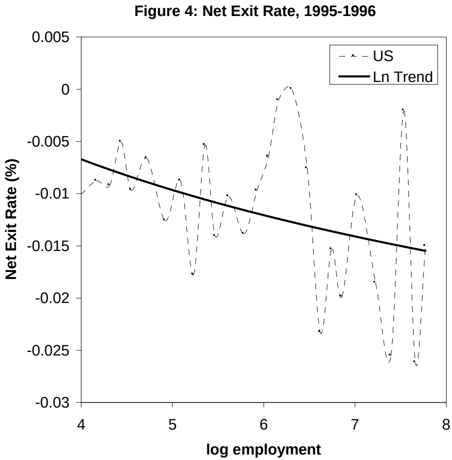
Facts on Establishments

- The size distribution of establishments has thinner tails than a Pareto distribution with coefficient one

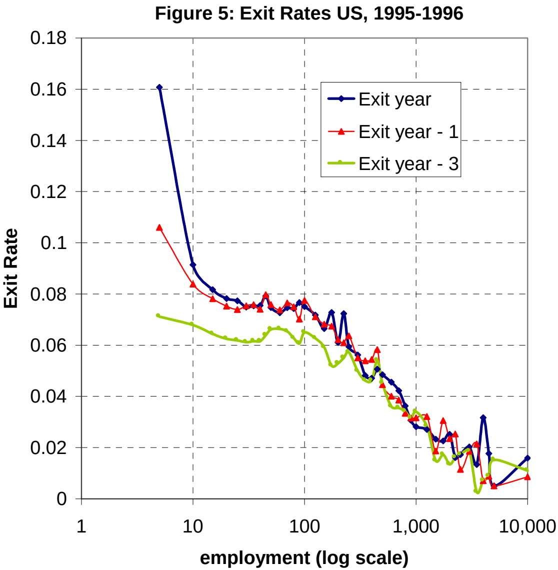


Facts on Establishments

- Small establishments exit (net) more than large establishments



Facts on Establishments: Not only selection



Key Elements

- We present a theory based on the accumulation of industry specific human capital
- The stock of specific human capital determines industry factor prices, which determines the size of the establishment
- The resulting industry production function exhibits diminishing returns, which leads to mean reversion in specific human capital
- As long as establishments respond monotonically to factor prices, this leads to scale dependence in growth rates
- Together with the degree of substitutability in consumption, this leads to a scale dependent net exit process
- These implications on growth and net exit rates lead to a size distribution with thinner tails than a Pareto distribution with coefficient one

Key Elements

- The importance of this mechanism depends on the degree of diminishing returns to industry specific human capital
  - If physical capital share is large, human capital share is small
  - If human capital share is small, the degree of diminishing returns to human capital is large
- Our theory predicts that, as we increase the physical capital share from zero, scale dependence should increase
- Using new data on both growth rates by size, and on size distributions, we show that:
  - scale dependence is larger in more capital intensive industries, and
  - sectoral differences in scale dependence are large

The Model: Households

- Order preferences over consumption according to

$$(1-\delta)E_0\left[\sum_{t=0}^{\infty}\delta^t N_t \ln\left(\frac{C_t}{N_t}\right)\right]$$

- Produce final consumption good from inputs of J other goods

$$C_t + \sum_{j=1}^{J} X_{tj} = B\prod_{j=1}^{J}\left(Y_{tj} - I_{tj}\right)^{\theta_j}.$$

- Accumulate industry specific physical and human capital according to

$$\begin{aligned} K_{t+1j} &= K_{tj}^{\lambda_j} X_{tj}^{1-\lambda_j} \\ H_{t+1j} &= A_{t+1j} H_{tj}^{\omega_j} I_{tj}^{1-\omega_j} \end{aligned}$$

- Grow at rate g_N, and $\sum_j N_{tj} \leq N$

The Model: Technology

- J goods produced in J *industries* which are grouped into *sectors*
  - Technology is identical within sectors, but productivity and stocks of industry-specific capital vary
  - All establishments within an industry are identical (later relax this)
- Establishments pay fixed cost F_j to operate (in units of the produced good)
- Establishments in operation hire labor n_{tj} and industry-j-specific physical, k_{tj}, and human, h_{tj}, capital to produce output according to

$$y_{tj} = \left[k_{tj}^{\alpha_j} \left(h_{tj}^{\beta_j} n_{tj}^{1-\beta_j} \right)^{1-\alpha_j} \right]^{\gamma_j}$$

with $\gamma_j < 1$

Social Optimum

- Without integer constraints, welfare theorems are satisfied
- Choose $\left\{C_{tj}, X_{tj}, I_{tj}, N_{tj}, \mu_{tj}, H_{tj}, K_{tj}\right\}_{t=0,j=1}^{\infty,J}$ to maximize

$$(1-\delta)E_0\left[\sum_{t=0}^{\infty}\delta^t N_t \ln\left(\frac{C_t}{N_t}\right)\right]$$

$$\text{s.t.} \quad C_t + \sum_{j=1}^{J} X_{tj} = B\prod_{j=1}^{J}\left(Y_{tj} - I_{tj}\right)^{\theta_j},$$

$$Y_{tj} + F_j\mu_{tj} = \left[K_{tj}^{\alpha_j}\left(H_{tj}^{\beta_j}N_{tj}^{1-\beta_j}\right)^{1-\alpha_j}\right]^{\gamma_j}\mu_{tj}^{1-\gamma_j},$$

$$K_{t+1j} = K_{tj}^{\lambda_j}X_{tj}^{1-\lambda_j} \quad \text{and} \quad H_{t+1j} = A_{t+1}H_{tj}^{\omega_j}I_{tj}^{1-\omega_j},$$

$$N_t = \sum_{j=1}^{J} N_{tj}$$

Establishment Sizes

- The problem of choosing the number of establishments is static. The first order condition for μ_{tj} is

$$F_j = \left(1-\gamma_j\right) y_{tj} = \left(1-\gamma_j\right) \left[\left(\frac{K_{tj}}{\mu_{tj}}\right)^{\alpha_j} \left(\left(\frac{H_{tj}}{\mu_{tj}}\right)^{\beta_j} \left(\frac{N_{tj}}{\mu_{tj}}\right)^{1-\beta_j} \right)^{1-\alpha_j} \right]^{\gamma_j}$$

- The resource constraint becomes

$$Y_{tj} \leq \gamma_j \left[\frac{1-\gamma_j}{F_j}\right]^{\frac{1-\gamma_j}{\gamma_j}} K_{tj}^{\alpha_j} \left(H_{tj}^{\beta_j} N_{tj}^{1-\beta_j}\right)^{1-\alpha_j}$$

- TFP in an industry depends on factor shares and fixed costs
- Industries face a constant returns to scale production function
- This yields a standard growth model consistent with balanced growth

Establishment Sizes

- Establishment size in industry j is then given by

$$n_{tj} = \frac{N_{tj}}{\mu_{tj}} = \left[\frac{F_j}{1-\gamma_j}\right]^{\frac{1}{\gamma_j}} \left(\frac{N_{tj}}{K_{tj}}\right)^{\alpha_j} \left(\frac{N_{tj}}{H_{tj}}\right)^{\beta_j\left(1-\alpha_j\right)}$$

- So establishment growth rates satisfy

$$\begin{aligned} \ln n_{t+1j} - \ln n_{tj} &= \Big(\alpha_j + \beta_j\left(1-\alpha_j\right)\Big) g_N - \alpha_j \left[\ln K_{t+1j} - \ln K_{tj}\right] \\ &\quad -\beta_j\left(1-\alpha_j\right)\left[\ln H_{t+1j} - \ln H_{tj}\right], \end{aligned}$$

- We mostly abstract from population growth, and assume aggregate economy is in steady state

Establishment Growth Rates

- To begin, when do we get scale *independent* growth?
- If output in an industry has no effect on the pace of its human capital accumulation
  - If we eliminate human capital as a factor of production ($\left(1-\alpha_j\right)$ or β_j equal zero), establishment growth is deterministic constant (unless scale variance of A_{tj})
  - If human capital is accumulated exogenously (limit as $\omega_j \to 1$)
- If β_j, $\left(1-\alpha_j\right)$, $\omega_j > 0$, get scale *dependent* growth

$$\ln n_{t+1j} - \ln n_{tj} = n^C - \left(1-\omega_j\right)\left(1-\beta_j+\alpha_j\beta_j\right)\ln n_{tj} - \beta_j\left(1-\alpha_j\right)\ln A_{t+1j}$$

Proposition:

- *Establishment growth rates are weakly decreasing in size*
- *The higher is the physical capital share, the faster growth rates decline with size*
- *The growth rate of establishments is independent of its size only if either human capital is not a factor of production or human capital evolves exogenously*

Corollary: *Same is true for net exit rates*

Size Distribution

Proposition: *(Zipf's Law) If either human capital is not a factor of production, or human capital evolves exogenously, the size distribution of establishments converges to a Pareto distribution with shape coefficient one*

Proposition: *(Thinner Tails) For any* $\alpha_j, \beta_j, \omega_j \in (0, 1)$, *the invariant distribution of establishment sizes has thinner tails than the Pareto distribution with coefficient one. Other things equal, if* $\alpha_j > \alpha_k$, *the invariant distribution of establishments in sector* j *has thinner tails than the invariant distribution of establishments in sector* k.

- Thinner tails manifest as concave log rank - log size plots

Digression: Gabaix (1999)

- **Proposition**: Suppose there are J types of sectors, each with parameters satisfying the conditions above (and hence with sectors satisfying Zipf's Law). Then the entire establishment size distribution satisfies Zipf's Law.
- Sketch of Proof: Let λ_j be the proportion of type j establishments. For each industry j

$$P\left(n > N | \text{type } j\right) \propto \frac{A_j}{N}.$$

Hence

$$P\left(n > N\right) = \sum_{j=1}^{J} P\left(n > N | \text{type } j\right) \lambda_j \propto \frac{A}{N} = \frac{\sum_{j=1}^{J} \lambda_j A_j}{N}.$$

Digression: Gabaix (1999)

- **Proposition**: Suppose that establishment sizes n_t are determined by Gibrat's Law $n_{t+1} = \gamma_{t+1} n_t$, for some γ_t iid with distribution $f(\gamma)$. Then there exists an invariant distribution of establishment sizes satisfying Zipf's Law
- Sketch of Proof: Normalize establishment sizes so that average size stays constant; then normalized growth rates satisfy $E[\gamma] = 1$. Then

$$\begin{aligned} G_{t+1}(N) &= P(n_{t+1} > N) = P\left(\gamma_{t+1} n_t > N\right) \\ &= E\left[1_{n_t > N/\gamma_{t+1}}\right] = E\left[G_t\left(\frac{N}{\gamma_{t+1}}\right)\right] = \int_0^\infty G_t\left(\frac{N}{\gamma}\right) f(\gamma)\, d\gamma. \end{aligned}$$

- If there exists an invariant distribution G, we must have

$$G(N) = \int_0^\infty G\left(\frac{N}{\gamma}\right) f(\gamma)\, d\gamma,$$

which is obviously satisfied by a distribution of the form $G(N) = a/N$.

Robustness

- Robust to:
    - Establishment heterogeneity
    - Establishment costs
    - Market structure: monopolistic competition
    - Human capital accumulated by learning by doing

Robustness: Establishment Heterogeneity

- So far, we have abstracted from heterogeneity among establishments *within* an industry in order to focus on heterogeneity *across* industries
- Assume that, after deciding to produce in a period, each establishment i receives a mean one i.i.d. shock z_i
- Within an industry, relative establishment sizes are then given by

$$\frac{n_i}{n_j} = \left(\frac{z_i}{z_j}\right)^{\frac{1}{1-\gamma}}$$

- The shock has no effect on the *mean* growth and net exit rates in an industry, and therefore in a sector. Nor does it affect the relationship between factor intensities and establishment dynamics.
- In this case, Zipf's Law will hold under the same conditions if the distribution within an industry is also Pareto with coefficient one.

Robustness: Establishment Heterogeneity

- Assume that hiring n_{tj} workers entails a managerial cost of $F_j n_{tj}^{\xi_j}$ for $\xi_j < 1$ so the establishment problem is

$$\max_{k_{tj},h_{tj},n_{tj}} \Pi \equiv \max_{k_{tj},h_{tj},n_{tj}} y_{tj} - r_{tj}k_{tj} - s_{tj}h_{tj} - w_{tj}n_{tj} - F_j n_{tj}^{\xi_j}.$$

- This implies a establishment size given by

$$n_{tj} = \frac{N_{tj}}{\mu_{tj}} = \left[\frac{\left(1-\gamma_j\right)}{\left(1-\xi_j\right)F_j}\right]^{\frac{1}{\xi_j-\gamma_j}} \left(\frac{N_{tj}}{K_{tj}}\right)^{\frac{\alpha_j\gamma_j}{\gamma_j-\xi_j}} \left(\frac{N_{tj}}{H_{tj}}\right)^{\frac{\beta_j\left(1-\alpha_j\right)\gamma_j}{\gamma_j-\xi_j}}.$$

- The only difference is that both employment and output will respond to changes in factor supplies
  - For $\xi_j < \gamma_j$, as before, higher specific factor stocks lead to smaller establishment sizes
  - For $\xi_j > \gamma_j$, higher specific factor stocks lead to larger establishment sizes

Robustness: Market Structure

- Key for mechanism to work is that intensive margin (establishment size) and extensive margin (establishment net exit) must both operate
- Now each industry consists of a continuum of potential varieties which we index by ϖ. Physical and human capital are industry-specific (but *not* variety-specific)
- Output of each variety D_{tj}^{ϖ} is combined by the household using a constant elasticity of substitution production function with parameter σ_j to produce a composite good for each industry
- Together, they produce an aggregate good that is used for both final consumption and investment
- A households demand for a variety ϖ in industry j is

$$D_{tj}^{\varpi}\left(p_{tj}^{\varpi}\right) = E_{tj}^{\varpi} \frac{\left(p_{tj}^{\varpi}\right)^{-\sigma_j}}{\int\limits_{0 \leq \varpi \leq \Omega_{tj}} \left(p_{tj}^{\varpi}\right)^{1-\sigma_j} d\varpi},$$

Robustness: Market Structure

- Establishments pay fixed costs, F_j, to produce variety ϖ using a constant returns to scale Cobb-Douglas technology in labor and physical capital
- The constant markup plus zero profits from free entry imply

$$D_{tj}^{\varpi}\left(p_{tj}^{\varpi}\right) = F_j\left(\sigma_j - 1\right)$$

- The size of establishments is

$$n_{tj}^{\varpi} = F_j\sigma_j\left(\frac{N_{tj}}{K_{tj}}\right)^{\alpha_j}\left(\frac{N_{tj}}{H_{th}}\right)^{\beta_j\left(1-\alpha_j\right)}$$

Robustness: Learning-by-Doing Externalities

- Suppose human capital is accumulated according to

$$H_{t+1j} = A_{t+1j} H_{tj}^{\omega_j} Y_{tj}^{1-\omega_j},$$

- Production occurs according to

$$Y_{tj} + F_j \mu_{tj} = \left[K_{tj}^{\alpha_j} \left(H_{tj} N_{tj} \right)^{1-\alpha_j} \right]^{\gamma_j} \mu_{tj}^{1-\gamma_j},$$

so human capital operates exactly like labor augmenting technological progress.

- Use a pseudo-planner problem to show

$$\ln n_{t+1} - \ln n_t = n^C - \alpha_j \left(1 - \omega_j \right) \ln n_t - \left(1 - \alpha_j \right) \ln A_{t+1},$$

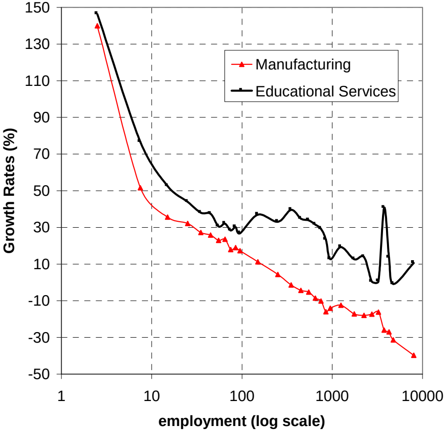
Implications of Theory

- Our theory implies a positive relationship between the degree of diminishing returns to industry specific human capital and scale dependence
- If physical capital shares are larger, the degree of diminishing returns to human capital is larger
- We should observe a positive relationship between physical capital shares and
  1. the rate at which establishment growth rates decline with size
  2. the thinness of the tails of the establishment size distribution
  3. the rate at which net exit decreases with size
- Compare Manufacturing with a capital share of .322 with Educational Services with a capital share of .054

Growth Rates and Capital Shares: Two Sectors

- Even though small establishments grow at similar rates, there are large differences across industries for large establishments

Figure 6: Establishment Conditional Growth Rates by Sector, 1990-2000



Growth Rates and Capital Shares: Many Sectors

- We use new establishment growth data from BITS by very fine size categories at the 2 digit SIC code level
- Physical capital shares are calculated as 1 minus labor shares from the BEA, and we also adjust for the share of value added
- We run the regression using GLS

$$\ln\left(\frac{n_{t+1j}}{n_{tj}}\right) = \tilde{a}_j + \tilde{b}\ln n_{tj} + \tilde{e}\alpha_j \ln n_{tj} + \tilde{\varepsilon}_{tj},$$

- This amounts to fitting an exponential trend where the parameter varies linearly with capital shares by sector

	1990-2000			
	Var. $= 1/\mu_j$		Var. $= \left(1-\alpha_j\right)^2/\mu_j$	
		(adjusted)		(adjusted)
$\tilde{e}$	-0.1115	-0.1517	-0.1488	-0.1814
Standard error	0.0255	0.0314	0.0304	0.0325
p v.	0.0000	0.0000	0.0000	0.0000

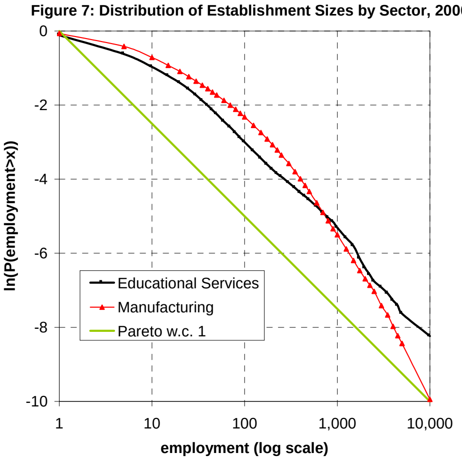
Manufacturing vs. Non-Manufacturing

- The last ten years have witnessed a substantial decline in employment among large manufacturing establishments
- Could this be driving the larger scale dependence observed in these sectors?
- We replicate the previous exercise for non-manufacturing and manufacturing sectors separately

	Var. $= 1/\mu_j$				Var. $= (1-\alpha_j)^2/\mu_j$			
	Manufacturing		Non-Manufacturing		Manufacturing		Non-Manufacturing	
		(adj.)		(adj.)		(adj.)		(adj.)
$\tilde{e}$	-0.0524	-0.0485	-0.1159	-0.1619	-0.0876	-0.0720	-0.1556	-0.1922
s.e.	0.0981	0.1213	0.0265	0.0329	0.0972	0.1295	0.0322	0.0342
p v.	0.5930	0.6900	0.0000	0.0000	0.3680	0.5780	0.0000	0.0000

Firm Size and Capital Shares: Two Sectors

- For both distributions to match it would be necessary to reallocate a large proportion of workers



Firm Size and Capital Shares: Many Sectors

- We use new data on the size distribution on establishments from SUSB
  - Small establishment size categories
  - All non-farm private sectors
  - For establishments
- For each sector we use OLS to estimate

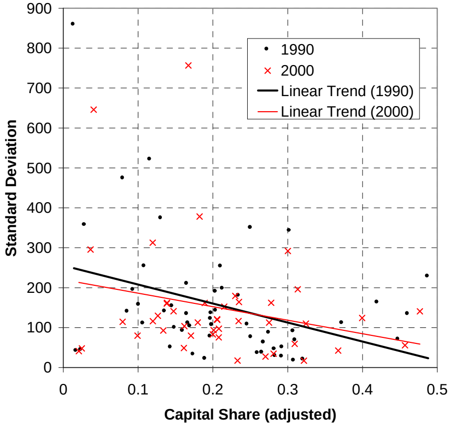
$$\ln P_j = \hat{a}_j + \hat{b}_j \ln n_j + \hat{d} \left(\ln n_j\right)^2 + \hat{e}\alpha_j \left(\ln n_j\right)^2 + \hat{\varepsilon}_{tj}$$

	1990		2000	
		(adj.)		(adj.)
$\hat{e}$	-0.1015	-0.0402	-0.0730	-0.1309
s.e.	0.0152	0.0145	0.0167	0.0163
p v.	0.0000	0.0060	0.0000	0.0000

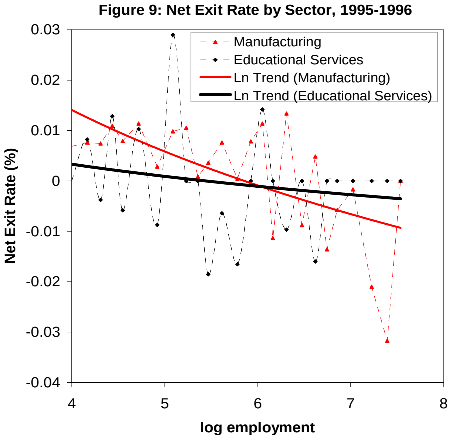
Variance and Capital Shares: Many Sectors

- Variance of establishment sizes within a sector decrease with α_j as in the theory

Figure 10: SD of Establishment Sizes and Capital Shares, 1990 and 2000



Net Exit Rates and Capital Shares: Two Sectors



Net Exit Rates and Capital Shares: Many Sectors

- We focus on the size distribution of net exit when establishments exit/enter and one year before/after they exit/enter
- We run the following regression using weighted least squares
- We use the equation implied by the model
  - Results biased down if industry employment reacts to shocks

$$\ln\left(1 + NER_j\right) = \check{a}_j + \check{b}\ln n_j + \check{e}\alpha_j \ln n_j + \check{\varepsilon}_{tj},$$

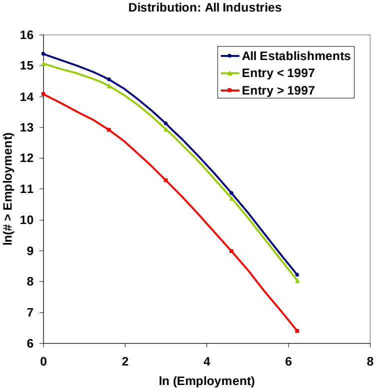
	Var. $= 1/\check{\mu}_j$				Var. $= \left(1-\alpha_j\right)^2/\check{\mu}_j$			
	Size in 1995-1996	(adj.)	Size in 1994-1997	(adj.)	Size in 1995-1996	(adj.)	Size in 1994-1997	(adj.)
$\check{e}$	-0.0314	-0.0331	-0.0172	-0.0186	-0.0324	-0.0280	-0.0164	-0.0151
s.e.	0.0029	0.0034	0.0024	0.0028	0.0036	0.0036	0.0029	0.0030
p v.	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Share and Depreciation of Human Capital

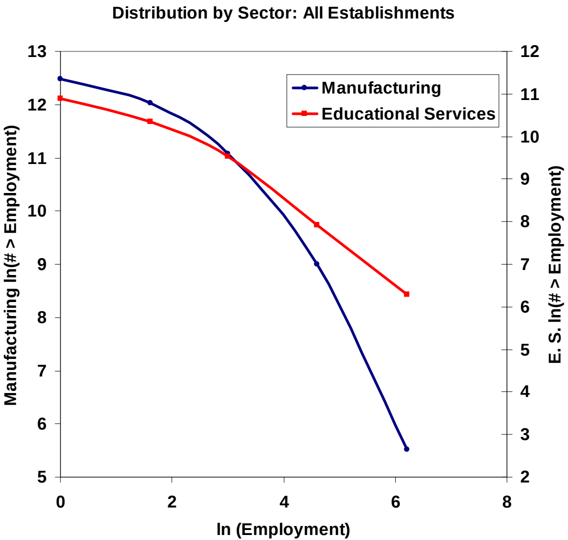
- Our estimation of $\tilde{b}$ and $\tilde{e}$ assumes that both β_j and ω_j are constant across industries
- From our estimates we can obtain average values of β_j and ω_j
- Implied share of specific human capital in labor services (β) between .432 and .556
- Implied share of investments in human capital production ($1 - \omega$) between .258 and .326
  - Similar to a ten year depreciation rate of human capital

Age Effects

- What is the role of age effects on these results?
- Lack of data prevents us from controlling for age, but age effects die out too fast to account for findings

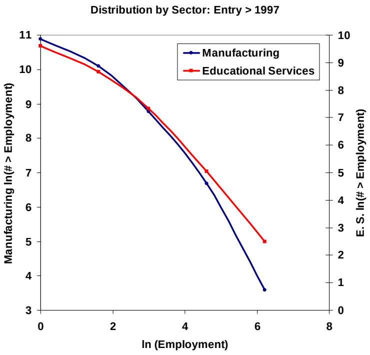


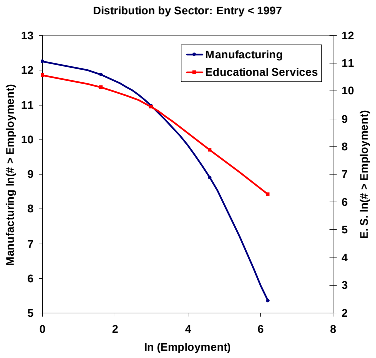
Age Effects


Distribution by Sector: All Establishments
Manufacturing
Educational Services
Manufacturing ln(# > Employment)
E. S. ln(# > Employment)
ln (Employment)
13
12
11
10
9
8
7
6
5
12
11
10
9
8
7
6
5
4
3
2
0
2
4
6
8

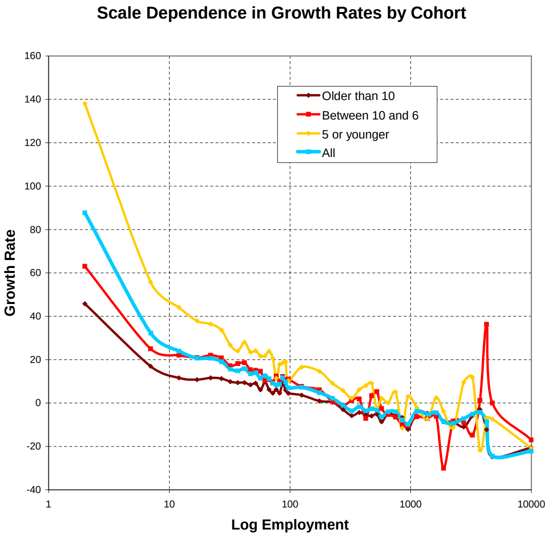
Age Effects

- Controlling for age does not make effect disappear
- After 5 years age effects are hard to see



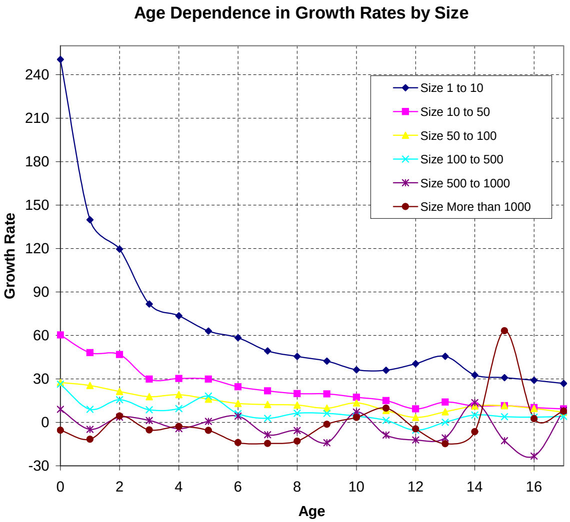


Age Effects



Scale Dependence in Growth Rates by Cohort
Older than 10
Between 10 and 6
5 or younger
All
Growth Rate
Log Employment
160
140
120
100
80
60
40
20
0
-20
-40
1
10
100
1000
10000

Age Effects


Age Dependence in Growth Rates by Size
Growth Rate
Age
240
210
180
150
120
90
60
30
0
-30
0
2
4
6
8
10
12
14
16
Size 1 to 10
Size 10 to 50
Size 50 to 100
Size 100 to 500
Size 500 to 1000
Size More than 1000